NONLINEAR PHENOMENA IN CLOSED FLOWS NEAR CRITICAL REYNOLDS NUMBERS

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It is shown that small normal perturbations in a closed steady fluid flow always either decrease or increase monotonically irrespective of the form of the walls or the character of the motion. Making use of this result, steady motions near critical Reynolds number R are investigated in general form. In the general case two series of steady motions, analytically dependent on R, intersect at the critical R. The motions of one series, starting from equilibrium at $R = 0$, are stable for R less than a critical value R_0, and are unstable for $R > R_0$. The motions of the second series, on the contrary, are stable above some critical point, are unstable below it and do not exist at a certain $R_1 < R_0$. The motions of both series coincide at the critical point itself, and near R_0 their difference varies as $(R - R_0)$. A special case may occur if a problem permits a symmetric transformation (for example, an arbitrary displacement along an axis) and the perturbation which disturbs the stability is invariant with respect to this transformation. Two new series of steady motions, analytically dependent on $(R - R_0)^{1/2}$, then appear above the critical point. The situation is exactly the same in the case of a fluid moving between two rotating cylinders.

Until recently in hydrodynamic stability theory, the stability of non-closed flows was studied almost exclusively, efforts being directed mainly to the calculation of the critical Reynolds number, i.e. to the solution of the linear small perturbation equations. Only Landau [1,2] has raised the question of phenomena at Reynolds numbers slightly in excess of the critical, and has shown that there must exist (he had in mind non-closed flows) unsteady periodic motion, the amplitude of which is proportional to $(R - R_0)^{1/2}$. As regards closed flows, apparently only the motion of a fluid between two rotating cylinders (Taylor's problem) has been investigated. Taylor [3] calculated the critical Reynolds number for this problem and the perturbation which breaks down the steady flow, and it was then shown in his experiments [3] (cf. also the work of Lewis [4]) and in the experiments of Stuart that all the theoretical conclusions are correct. In his experiments it is also seen that after the break-down of the basic flow a new steady motion is established, whose intensity differs from the intensity of the basic flow by an amount which is proportional to $(R - R_0)^{1/2}$. Stuart applied the Landau concept to the Taylor problem and showed that, although the motion here is closed, the conclusions of Landau remain partially in effect and the theory is in excellent quantitative agreement with experiment. The impression generated is that the laws indicated by Landau must also hold for closed flows.

The Taylor case, however, is not a typical case of closed flow. The length of the cylinders in the Taylor experiments was 800 times greater than the width of the space filled with fluid, and consequently it may be thought that the phenomena observed there should be like the phenomena in infinite non-closed flows. It would be very interesting to investigate experimentally some typical closed motion, for example the motion between two rotating spherical surfaces.

In this paper, a general investigation of the nonlinear hydrodynamic equations is carried out for closed flows near critical Reynolds numbers, and it is shown that the Taylor problem is really a special case and that phenomena near critical points in typical closed flows look absolutely different. The method used here is a development of the method of [6].

1. Normal perturbations. A fluid filling the volume (V), whose walls (S) are moving steadily with velocities $\mathbf{U}_s$ which differ at different points, is considered. The walls may consist of several parts having the form of a body of revolution, but more complex cases, for example when the walls are made of a flexible ribbon moving parallel to "itself", can also be considered. It is assumed that a steady fluid motion whose stability should be investigated is possible under these conditions.

We introduce a characteristic length dimension l, a velocity ν/l and a time l^2/ν (ν is the kinematic viscosity), and also a Reynolds number

$$R = \frac{lU_s^{(0)}}{\nu} \tag{1.1}$$

where $U_s^{(0)}$ is a characteristic wall velocity; the equations of motion will be

$$\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p - \operatorname{rot}\operatorname{rot}\mathbf{v}, \qquad \operatorname{div}\mathbf{v} = 0, \qquad \mathbf{v}|_s = R\mathbf{U}_s \tag{1.2}$$

Here $\mathbf{U}_s$ designates the velocity distribution on the walls, normalized with a single characteristic velocity.

Steady fluid flows will then be investigated for a given form of the velocity distribution on the walls and for various Reynolds numbers. Such steady flows satisfy the equations

$$\nabla P + \operatorname{rot}\operatorname{rot}\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} = 0, \qquad \operatorname{div}\mathbf{U} = 0, \qquad \mathbf{U}|_s = R\mathbf{U}_s \tag{1.3}$$

To investigate their stability, we shall consider the perturbation flow

$$\{\mathbf{v},\ p'\} = \{\mathbf{U},\ P\} + \{\mathbf{u},\ p\} \tag{1.4}$$

If this is substituted into (1.2) and if (1.3) is taken into account, and if subsequently, considering the perturbation to be small, quadratic terms in the perturbation are neglected and

$$\{\mathbf{u},\ p\} \sim e^{-\lambda t} \tag{1.5}$$

is assumed, the linear equations for the normal perturbations of the steady motion $\mathbf{U}(R)$

$$-\lambda\mathbf{u} + L[\mathbf{u};\mathbf{U}(R)] = -\lambda\mathbf{u} + \nabla p + \operatorname{rot}\operatorname{rot}\mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{U} = 0$$
$$\operatorname{div}\mathbf{u} = 0, \qquad \mathbf{u}|_s = 0 \tag{1.6}$$

are then obtained.

For simplicity we will consider the eigenvalues λ to be prime numbers*. The solution of problem (1.6) then gives (for a finite volume) an infinite sequence of normal perturbations and the decrements that correspond to them

$$\mathbf{u}_\alpha,\quad p_\alpha;\quad \lambda_\alpha \qquad (\alpha = 0, 1, 2\ldots) \tag{1.7}$$

* The case of divisible numbers λ is of no particular interest, since in this problem "adjoint" perturbations which vary with time as $(1 + at)e^{-\lambda t}$, etc. will not be possible (they are possible, in principle, for non-self-conjugate L). The fact is that the operator L transforms analytically into a self-conjugate form as $R \to 0$.

Here, the letter L in the symbol $L[\phi;\ \chi]$ designates an operator which acts on the function-argument ϕ, and χ is a function on which L depends as a parameter.

numbered in the order of increasing real parts of the number λ. Physically it is clear that this sequence will be complete: any small perturbation $\mathbf{u}$ must be made up of normal perturbations which vary exponentially with time.

Because problem (1.6) is non-self-conjugate, the normal perturbations are non-orthogonal among themselves and the decrements may be complex. A problem, conjugate with (1.6), is obtained if the equations which are complex conjugates of (1.6) are multiplied by the variable vector $\mathbf{v}$ and integrated over the volume, and if the derivatives are interchanged from $\mathbf{u}^*$ to $\mathbf{v}$ using the Gauss theorem, and the factor multiplying $\mathbf{u}^*$ is set equal to zero. There is then obtained

$$-\lambda^*\mathbf{v} + L^+[\mathbf{v};\ \mathbf{U}(R)] = -\lambda^*\mathbf{v} + \nabla q + \operatorname{rot}\operatorname{rot}\mathbf{v} - (\mathbf{U}\cdot\nabla)\mathbf{v} + \nabla(\mathbf{U}\cdot\mathbf{v}) = 0$$

$$\operatorname{div}\mathbf{v} = 0, \qquad \mathbf{v}|_s = 0 \tag{1.8}$$

Problem (1.8) also has a complete sequence of solutions (conjugate normal perturbations)

$$\mathbf{v}_\alpha,\ \ q_\alpha,\ \ \lambda_\alpha^* \qquad (\alpha = 0, 1, 2\ldots) \tag{1.9}$$

whose decrements are complex conjugates of the decrements (1.7). These solutions do not have a straightforward physical meaning, but they are orthogonal to the normal perturbations (1.7) and are necessary to determine the coefficients of the expansion of an arbitrary perturbation in terms of the normal perturbations. Indeed, from (1.6) and (1.8) we obtain

$$(\lambda_\beta - \lambda_\alpha)\int \mathbf{v}_\beta^* \cdot \mathbf{u}_\alpha dV = \int \{\mathbf{v}_\beta^* L[\mathbf{u}_\alpha] - L^+[\mathbf{v}_\beta^*]\,\mathbf{u}_\alpha\}\, dV = 0$$

Hence, with appropriate normalization

$$\int \mathbf{v}_\beta^* \cdot \mathbf{u}_\alpha dV = \delta_{\beta\alpha} \tag{1.10}$$

Every incompressible flow which vanishes at the walls must be expanded in a series of the form

$$\{\mathbf{u},\ p\} = \sum_\alpha a_\alpha \{\mathbf{u}_\alpha,\ p_\alpha\}, \qquad a_\alpha = \int \mathbf{v}_\alpha^* \mathbf{u}\, dV \tag{1.11}$$

Investigation of the stability reduces to calculating the decrements λ. The steady motion is stable with respect to the normal perturbation (α) if $\operatorname{Re}\lambda_\alpha > 0$.

2. The basic series of steady flows. For given geometric conditions and a given velocity distribution on the walls, there will exist a series of steady motions which vary continuously with Reynolds number and for $R = 0$, i.e. for stationary walls, reduce to equilibrium. It is easily shown that no steady motion whatever is possible for stationary walls except the basic one $\mathbf{U}(0) = 0$. Indeed, for $R = 0$ it follows from (1.2) that

$$\frac{\partial}{\partial t}\int \frac{\mathbf{v}^2}{2} dV = -\int (\operatorname{rot} \mathbf{v})^2 dV < 0$$

Thus, if the motion is steady, then

$$\operatorname{rot} \mathbf{v} = 0, \qquad \operatorname{div} \mathbf{v} = 0, \qquad \mathbf{v}|_s = 0$$

and, consequently, $\mathbf{v} = 0$. This argument even demonstrates that the "motion" $\mathbf{U} = 0$ is stable, which also follows from Equations (1.6). For $\mathbf{U} = 0$ these take the form

$$-\lambda \mathbf{u} + \nabla p + \operatorname{rot}\operatorname{rot} \mathbf{u} = 0, \qquad \operatorname{div} \mathbf{u} = 0, \qquad \mathbf{u}|_s = 0 \tag{2.1}$$

This boundary-value problem is self-conjugate so that its solutions

$$\mathbf{u}_\alpha(0), \qquad p_\alpha(0); \qquad \lambda_\alpha(0) \tag{2.2}$$

are real and

$$\lambda_\alpha(0)\int \mathbf{u}_\alpha^2 dV = \int (\operatorname{rot} \mathbf{u}_\alpha)^2 dV > 0$$

For small R the motions of the basic series can be represented in the form of a series in R

$$\begin{Bmatrix} \mathbf{U}\,(R) \\ P\,(R) \end{Bmatrix} = R \begin{Bmatrix} \mathbf{U}_1 \\ P_1 \end{Bmatrix} + R^2 \begin{Bmatrix} \mathbf{U}_2 \\ P_2 \end{Bmatrix} + \ldots \tag{2.3}$$

Substituting in (1.3) gives the equations

$$\nabla P_1 + \operatorname{rot}\operatorname{rot} \mathbf{U}_1 = 0, \qquad \operatorname{div} \mathbf{U}_1 = 0, \qquad \mathbf{U}_1|_s = \mathbf{U}_s$$
$$\nabla P_n + \operatorname{rot}\operatorname{rot} \mathbf{U}_n = -\sum_{k=1}^{n-1} (\mathbf{U}_k \cdot \nabla)\, \mathbf{U}_{n-k}$$
$$\operatorname{div} \mathbf{U}_n = 0, \qquad \mathbf{U}_n|_s = 0 \qquad (n > 1) \tag{2.4}$$

solving which permits all of the $\mathbf{U}_n$ to be determined successively. The series (2.3) can be analytically continued to the first singular point lying on the real axis R. It will be further assumed that either there is no such point at all or it lies at a very large value of R, so that in the whole region of Reynolds numbers of present interest the motions of the basic series will exist. It will be possible later to obtain some information on the singular point at which the basic steady motions cease to exist.

For each motion of the basic series its own normal perturbations and its own decrements

$$\mathbf{u}_\alpha(R), \quad p_\alpha(R), \quad \lambda_\alpha(R) \tag{2.5}$$

will exist.

We shall show that all of these perturbations and their decrements are real. Let the realness already be proved for some Reynolds number R. For $R + \xi$ we shall assume

$$\begin{aligned} \lambda_\alpha(R+\xi) &= \lambda_\alpha + \xi\lambda^{(1)} + \xi^2\lambda^{(2)} + \dots \\ \mathbf{u}_\alpha(R+\xi) &= \mathbf{u}_\alpha + \xi\mathbf{u}^{(1)} + \xi^2\mathbf{u}^{(2)} + \dots \end{aligned} \tag{2.6}$$

and an analogous expression for the pressure. The basic motion for $R + \xi$ will be

$$\mathbf{U}(R+\xi) = \mathbf{U}(R) + \xi\mathbf{U}^{(1)} + \xi^2\mathbf{U}^{(2)} + \dots \tag{2.7}$$

Substituting in (1.6) gives, for terms containing ξ^n

$$\begin{aligned} \lambda_\alpha\mathbf{u}^{(n)} - L[\mathbf{u}^{(n)}\,\mathbf{U}(R)] + \lambda^{(n)}\mathbf{u}_\alpha = &-[\lambda^{(n-1)}\mathbf{u}^{(1)} + \dots + \lambda^{(1)}\mathbf{u}^{(n-1)}] - \\ &-[(\dot{\mathbf{U}}^{(n)}\cdot\nabla)\,\mathbf{u}_\alpha + \dots + (\mathbf{U}^{(1)}\cdot\nabla)\,\mathbf{u}^{(n-1)}] - \\ &-[(\mathbf{u}_\alpha\cdot\nabla)\,\mathbf{U}^{(n)} + \dots + (\mathbf{u}^{(n-1)}\cdot\nabla)\,\mathbf{U}^{(1)}] \end{aligned} \tag{2.8}$$

Multiplying this equality by the conjugate normal perturbation $\mathbf{v}_\alpha$ (which, by assumption, is real) and integrating, we shall obtain by virtue of (1.8)

$$\lambda^{(n)} = \int \mathbf{v}_\alpha\{\dots\}\,dV \tag{2.9}$$

The dots in the brackets here designate the right-hand side of Equation (2.8).

Each of the $\mathbf{u}^{(n)}$ can be expanded in terms of the perturbations (2.5):

$$\mathbf{u}^{(n)} = \sum_{\beta\neq\alpha} b_\beta^{(n)}\mathbf{u}_\beta(R) \tag{2.10}$$

(The exclusion of terms with $\beta = \alpha$ is equivalent to a change or normalization.) If such an expansion is substituted into (2.8), multiplied by some $\mathbf{v}_\beta$ for $\beta \neq \alpha$ and integrated, then there is obtained (2.11)

$$\begin{aligned} (\lambda_\beta - \lambda_\alpha)\,b_\beta^{(n)} = &\int \mathbf{v}_\beta\cdot(\mathbf{U}^{(n)}\cdot\nabla)\,\mathbf{u}_\alpha dV + \int \mathbf{v}_\beta\cdot(\mathbf{u}_\alpha\cdot\nabla)\,\mathbf{U}^{(n)}dV + \\ &+ \sum_{k=1}^{n-1}\lambda^{(k)}b_\beta^{(k)} + \sum_{k=1}^{n-1}\sum_{\gamma\neq\alpha}\left\{\int \mathbf{v}_\beta\cdot(\mathbf{U}^{(n-k)}\cdot\nabla)\,\mathbf{u}_\gamma dV + \int \mathbf{v}_\beta\cdot(\mathbf{u}_\gamma\cdot\nabla)\,\mathbf{U}^{(n-k)}dV\right\} b_\gamma^{(k)} \end{aligned}$$

From (2.9) and (2.11) it is seen that it is possible to determine successively all of the terms of the expansion (2.6) and all of them will be real. Because the normal perturbations are real for $R = 0$, they will be real throughout the entire region of existence of the basic steady motions. Consequently, if a fluid moves steadily in a closed volume and the motion belongs to the basic series, then any normal perturbation either decreases or increases monotonically for any Reynolds number.

3. The critical Reynolds number. It has been shown above that the basic steady motion for stationary walls (equilibrium) is stable. By continuation the motions of the basic series for sufficiently small R will also be stable. There exist cases for which the motion remains stable for all R, so that all of the $\lambda_\alpha(R)$ are always positive.

Cases are referred to here, for example, in which the walls move with an angular velocity, rotating about a common axis:

$$\mathbf{U}_s = \mathbf{n} \times \mathbf{r}, \qquad \mathbf{n} = \text{const}, \qquad \mathbf{n}^2 = 1 \tag{3.1}$$

In the basic steady motion the fluid will then rotate as a solid body, i.e.

$$\mathbf{U} = R \qquad \mathbf{n} \times \mathbf{r} \tag{3.2}$$

To investigate stability here it is simplest to transfer to a system of reference which rotates together with the walls. In Equations (1.6) it is then necessary to assume $\mathbf{U} = 0$ and to add the centrifugal force (which is the gradient of a scalar) and the Coriolis force. As a result we obtain

$$\begin{gathered} \lambda \mathbf{u} - \nabla f - \operatorname{rot} \operatorname{rot} \mathbf{u} + 2R\, \mathbf{u} \times \mathbf{n} = 0 \\ \operatorname{div} \mathbf{u} = 0, \qquad \mathbf{u}|_s = 0 \end{gathered} \tag{3.3}$$

Hence, the equality

$$\lambda \int |\mathbf{u}|^2 dV = \int |\operatorname{rot} \mathbf{u}|^2 dV - 2R\,\mathbf{n} \int \mathbf{u}^* \times \mathbf{u}\, dV$$

follows and consequently

$$\operatorname{Re} \lambda = \int |\operatorname{rot} \mathbf{u}|^2\, dV \Big/ \int |\mathbf{u}|^2 dV > 0 \tag{3.4}$$

Thus, solid rotation is stable for all R.

But it is also known that for a fluid which fills the space between two coaxial cylinders rotating with equal angular velocities there exists under certain conditions a critical Reynolds number R for which the least decrement vanishes. Although existing calculations are concerned with infinite cylinders, experiments with cylinders of finite length indicate that for some R_0 the motion does, in fact, become unstable.

Thus, there can exist series of steady motions which are stable for R less than a certain R_0 and unstable for $R > R_0$. We shall investigate steady motions near the critical point R_0 in which

$$\lambda_0(R_0) = 0, \qquad \lambda_\alpha(R_0) > 0 \qquad (\alpha > 0) \tag{3.5}$$

For simplicity we shall consider that all of the $\lambda_\alpha(R_0)$ are different.

4. The regular critical point. The critical point R_0 can be a regular (non-singular) point of the basic flow. We shall examine the expansion of the basic flow about some R, assuming for the present that R is not a critical point

$$\mathbf{U}(R+\xi) = \mathbf{U}(R) + \xi\mathbf{U}_1 + \ldots \tag{4.1}$$

The divergences of all terms of this expansion must be equal to zero, and at the walls it is necessary to have

$$\mathbf{U}(R+\xi)|_s = (R+\xi)\,\mathbf{U}_s \tag{4.2}$$

so that

$$\mathbf{U}_1|_s = \mathbf{U}_s \quad \mathbf{U}_n|_s = 0 \qquad (n > 1)$$

Series (4.1) must satisfy Equations (1.3), by virtue of which the sequence of boundary-value problems

$$L[\mathbf{U}_1] = 0, \qquad L[\mathbf{U}_n] = -\sum_{k=1}^{n-1}(\mathbf{U}_k\cdot\nabla)\,\mathbf{U}_{n-k} \equiv \mathbf{F}_n \qquad (n > 1) \tag{4.3}$$

is obtained.

To make the boundary conditions homogeneous for $n = 1$ also, we shall assume

$$\mathbf{U}_1 = R^{-1}\mathbf{U}(R) + \mathbf{U}_1' \tag{4.4}$$

Then

$$L[\mathbf{U}_1'] = -R^{-1}(\mathbf{U}\cdot\nabla)\,\mathbf{U} \equiv \mathbf{F}_1, \qquad \mathbf{U}_1'|_s = 0, \qquad \operatorname{div}\mathbf{U}_1' = 0 \tag{4.5}$$

Multiplying both sides of Equations (4.3) and (4.5) by one of the conjugate normal perturbations (1.9) and integrating, it is possible to obtain for any n the equality

$$\lambda_\alpha \int \mathbf{v}_\alpha\cdot\mathbf{U}_n dV = \int \mathbf{v}_\alpha\cdot\mathbf{F}_n dV \tag{4.6}$$

(For $n = 1$, $\mathbf{U}_n{}'$ should be on the left.) The integrals on the left sides of these equalities are the Fourier coefficients of the expansions of $\mathbf{U}_n$ in terms of $\mathbf{u}_\alpha$; therefore, if none of the decrements are equal to zero, then

$$\mathbf{U}_n = \delta_{n1} R^{-1}\mathbf{U}(R) + \sum_\alpha \mathbf{u}_\alpha \lambda_\alpha{}^{-1} \int \mathbf{v}_\alpha \cdot \mathbf{F}_n dV \tag{4.7}$$

The formulas lose their meaning if R is equal to the critical value R_0, because then $\lambda_0 = 0$. All of the rest of the decrements, however, are not equal to zero at the critical point and, consequently, all terms of the series (4.7), except those equal to zero, are continuous at R_0. Because the basic flow is also continuous at a regular point, it is clear that the zero term of the series (4.7) for $R = R_0$ will be

$$\lim_{R\to R_0} \frac{1}{\lambda_0(R)} \int \mathbf{v}_0(R)\cdot \mathbf{F}_n(R)\, dV \tag{4.8}$$

With this reservation, Formula (4.7) retains meaning at the critical point also, where, consequently

$$\lambda_0 = 0, \qquad \int \mathbf{v}_0 \cdot \mathbf{F}_n dV = 0 \qquad (n = 1, 2, \ldots) \tag{4.9}$$

For $n = 1$ relation (4.9) has the form

$$\int \mathbf{v}_0 \cdot (\mathbf{U}\cdot\nabla)\,\mathbf{U} dV = \int \operatorname{rot} \mathbf{v}_0 \,(d\mathbf{S} \times \mathbf{U}_s) = 0 \tag{4.10}$$

as is easily verified with the help of (1.3) and (1.8).

We shall show that near a regular critical point there exists a second series of steady motions for which this point is also not a singular point. The motions of both series coincide at the same critical point. We shall seek a second solution in the form

$$\begin{gathered}\mathbf{V}(R_0 + \xi) = \mathbf{U}(R_0 + \xi) + \varphi = \mathbf{U}(R_0) + \xi\,[\mathbf{U}_1(R_0) + \varphi_1] + \\ + \xi^2\,[\mathbf{U}_2(R_0) + \varphi_2] + \ldots \\ \operatorname{div}\varphi_n = 0, \qquad \varphi_n|_s = 0\end{gathered} \tag{4.11}$$

Substituting in equations of the form of (1.3) and taking (4.3) into consideration gives

$$L[\varphi_1] = 0 \tag{4.12}$$

$$L[\varphi_2] = -\{(\mathbf{U}_1\cdot\nabla)\,\varphi_1 + (\varphi_1\cdot\nabla)\,\mathbf{U}_1 + (\varphi_1\cdot\nabla)\,\varphi_1\} \equiv \mathbf{f}_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$L[\varphi_n] = -\sum_{k=1}^{n-1}\{(\mathbf{U}_k\cdot\nabla)\,\varphi_{n-k} + (\varphi_k\cdot\nabla)\,\mathbf{U}_{n-k} + (\varphi_k\cdot\nabla)\,\varphi_{n-k}\} \equiv \mathbf{f}_n$$

where the operator L is constructed from (1.6) with the help of $\mathbf{U}(R)$. From (4.12) it follows that

$$\varphi_1 = b_1 \mathbf{u}_0(R_0) \tag{4.13}$$

with an unknown constant b_1. For $n > 1$, by virtue of the characteristics of conjugate operators and the fact that at the critical point $\lambda_0 = 0$, we obtain

$$\int \mathbf{v}_0(R_0) L[\varphi_n] \, dV = \int L^+ [\mathbf{v}_0(R_0)] \cdot \varphi_n dV = 0$$

from which there follows the solvability condition for Equations (4.12)

$$\int \mathbf{v}_0 \cdot \mathbf{f}_n dV = 0 \tag{4.14}$$

For $n = 2$, after having substituted in $\mathbf{f}_2$ in place of ϕ_1 its value (4.13), we obtain from (4.14)

$$b_1{}^2 \int \mathbf{v}_0 \cdot (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 dV + b_1 \int \mathbf{v}_0 \cdot \{(\mathbf{U}_1 \cdot \nabla) \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla) \mathbf{U}_1\} \, dV = 0 \tag{4.15}$$

The choice of $b_1 = 0$, to be sure, leads again to the basic series; we shall take

$$b_1 = -\int \mathbf{v}_0 \{(\mathbf{U}_1 \cdot \nabla) \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla) \mathbf{U}_1\} \, dV \Big/ \int \mathbf{v}_0 \cdot (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 dV \qquad (4.16$$

which is possible if

$$\int \mathbf{v}_0 \cdot (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 dV \neq 0 \tag{4.17}$$

The singular case, in which this integral vanishes, will be investigated in Section 6.

For such a choice, Equation (4.12) will have a solution for $n = 2$ to which u_0 multiplied by an arbitrary constant b_2 must be added. It is easily seen that for $n = 3$ the constant b_2 enters linearly in the solvability condition (4.14) so that it is simply determined and will be real. After this, Equation (4.12) can be solved for $n = 3$, etc. Consequently, near a regular critical point a solution of the form

$$\mathbf{V}(R) = \mathbf{U}(R) + b_1 (R - R_0) \mathbf{u}_0 + \ldots \tag{4.18}$$

will exist.

The existence of such a solution beyond the critical point is not surprising; the basic steady motion there is unstable and, if a perturbation appears in it, a new steady motion (4.18) is eventually established.

It is stranger that the motions of the second series, as is evident from (4.18), are also possible below the critical point. Because the new motions are stable there, it is natural to think that the motions of the second series will be unstable for $R < R_0$. To show this, we shall write the equations for the normal perturbations of the second steady flow (4.18), which are analogous to Equations (1.6)

$$-\mu \mathbf{w} + L[\mathbf{w}, \mathbf{V}(R)] = -\mu \mathbf{w} + \nabla s + \operatorname{rot}\operatorname{rot}\mathbf{w} + (\mathbf{V}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{V} = 0$$
$$\operatorname{div}\mathbf{w} = 0, \qquad \mathbf{w}|_s = 0 \tag{4.19}$$

For small ξ, $\mathbf{V}$ can be replaced by its approximate value (4.18), after which we obtain

$$-\mu \mathbf{w} + L[\mathbf{w}, \mathbf{U}(R_0)] = -\xi[(\mathbf{U}_1\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{U}_1] - -\xi b_1[(\mathbf{u}_0\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{u}_0] \tag{4.20}$$

We shall calculate that μ which is close to zero, i.e. we shall find a $\mathbf{w}_0$ close to $\mathbf{u}_0$. For this we shall replace $\mathbf{w}$ by $\mathbf{u}_0$ in the right-hand side, and on the left-hand side we shall assume

$$\mathbf{w} = \mathbf{u}_0 + \sum_{\gamma\neq\alpha} a_\gamma \mathbf{u}_\gamma \tag{4.21}$$

where the a will be small quantities of first order. Then, after multiplying (4.20) by $\mathbf{v}_0(R_0)$ and integrating, we obtain to within ξ, taking (1.10) into consideration

$$\mu_0 = \xi\left\{2b_1\int \mathbf{v}_0\cdot(\mathbf{u}_0\cdot\nabla)\mathbf{u}_0 dV + \int \mathbf{v}_0\cdot[(\mathbf{U}_1\cdot\nabla)\mathbf{u}_0 + (\mathbf{u}_0\cdot\nabla)\mathbf{U}_1]\,dV\right\} \tag{4.22}$$

If the value of b_1 from (4.16) is substituted here, we finally obtain

$$\mu_0(R_0+\xi) = -\xi\int \mathbf{v}_0\cdot[(\mathbf{U}_1\cdot\nabla)\mathbf{u}_0 + (\mathbf{u}_0\cdot\nabla)\mathbf{U}_1]\,dV \tag{4.23}$$

This decrement must be compared to the decrement of the basic motion $\lambda_0(R_0+\xi)$, the calculation of which is carried out as above, except that it is necessary to set $b_1 = 0$ in (4.20) and (4.22). Thus

$$\lambda_0(R_0+\xi) = +\xi\int \mathbf{v}_0\cdot[(\mathbf{U}_1\cdot\nabla)\mathbf{u}_0 + (\mathbf{u}_0\cdot\nabla)\mathbf{U}_1]\,dV \tag{4.24}$$

and consequently

$$\mu_0(R_0+\xi) = -\lambda_0(R_0+\xi) + \dots \tag{4.25}$$

This means that where the basic motions are stable the motions of the second series are unstable and vice versa. Consequently, the intersection of two regular series of steady solutions occurs at a regular critical point with a change of stability.

5. The limiting critical point. The investigation which was conducted shows that the flows are stable above the critical point R_0 in all normal cases for which the integral (4.17) is not equal to zero, i.e. there will exist only steady motions of the second series which can also exist below the critical point, where they will be, however, unstable. In Section 2 it was shown that there are no steady motions whatsoever for $R = 0$ except the basic one $\mathbf{U} = 0$. Hence it follows that the motions of the second series must cease to exist for some $R_1 < R_0$.*

We shall show that this can occur if R_1 is a branch point of the second series of motions about which $\mathbf{V}(R)$ is expanded in powers of

$$\eta = (R - R_1)^{1/2} \tag{5.1}$$

For $R < R_1$ such a solution becomes imaginary, i.e. it ceases to exist physically. Let

$$\mathbf{V}(R_1 + \eta^2) = \mathbf{V}_0 + \eta\mathbf{V}_1 + \eta^2\mathbf{V}_2 + \dots \tag{5.2}$$

(and analogously for the pressure). It is obvious that

$$\operatorname{div}\mathbf{V}_n = 0, \quad \mathbf{V}_0|_s = R_1\mathbf{U}_s, \quad \mathbf{V}_2|_s = \mathbf{U}_s, \quad \mathbf{V}_n|_s = 0, \quad n \neq 2 \tag{5.3}$$

The equation of form (1.3) for terms which contain η gives

$$L(\mathbf{V}_1;\ \mathbf{V}_0) = 0 \tag{5.4}$$

This means that one of the decrements must be zero at the limiting critical point. In our case it can be either $\mu_0(R_1) = 0$ or $\mu_1(R_1) = 0$. Assuming the latter for definiteness, we obtain

$$\mathbf{V}_1 = a_1\mathbf{u}_1(R_1) \tag{5.5}$$

where a_1 is for the present an unknown constant.

Further, for terms which contain η^2, we obtain

$$L(\mathbf{V}_2;\ \mathbf{V}_0) = -(\mathbf{V}_1\cdot\nabla)\mathbf{V}_1 \tag{5.6}$$

If we assume here that $\mathbf{V}_2 = R_1^{-1}\mathbf{V}_0 + \mathbf{V}_2'$, there is obtained a boundary-value problem with the homogeneous boundary conditions

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* Cf. the observation of Landau in [2, Sect. 27].

$$L[\mathbf{V}_2', \mathbf{V}_0] = -R_1^{-1}(\mathbf{V}_0 \cdot \nabla)\mathbf{V}_0 - (\mathbf{V}_1 \cdot \nabla)\mathbf{V}_1, \qquad \mathbf{V}_2'|_s = 0 \qquad (5.7)$$

It is solvable only when its right-hand side is orthogonal to the conjugate perturbation $\mathbf{v}_1(R_1)$, i.e. when

$$a_1^2 R_1 \int \mathbf{v}_1 \cdot (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 dV = -\int \mathbf{v}_1 \cdot (\mathbf{V}_0 \cdot \nabla)\mathbf{V}_0 dV \qquad (5.8)$$

The integral on the right-hand side here is not equal to zero, because otherwise a solution expanded in powers of $(R - R_1)$ would exist. Thus, from (5.8) it is possible to determine that $a_1 \neq 0$ and near R_1 motions of the second series will have the form

$$\mathbf{V} = \mathbf{V}_0 + (R - R_1)^{1/2} a_1 \mathbf{u}_1(R_1) + \ldots \qquad (5.9)$$

We shall not examine the rest of the terms of this expansion.

Without further investigation it is not possible to exclude the possibility that the basic series of steady motions can also have a limiting point for some R_2, above which there will be no steady motions.

6. The branch point. A singular case is obtained when, for $\lambda_0 = 0$

$$\int \mathbf{v}_0 (\mathbf{u}_0 \nabla)\mathbf{u}_0 dV = 0 \qquad (6.1)$$

From Formulas (4.16) and (5.8) it is seen that R_0 can be neither a regular critical point nor a limiting point. The integral (6.1) can turn out to be zero either by chance (this possibility is excluded) or by virtue of the symmetry of the problem; just such a solution occurs in the case of the motion of fluid between two infinite cylinders. Here, the symmetry of the problem permits any displacement along the axis of the cylinders, and therefore the normal perturbations $\mathbf{u}_0$ and $\mathbf{v}_0$ depend on the coordinate z as $\cos kz$ or $\sin kz$, where z is measured along the axis of the cylinders. Each of the summed terms in the integral (6.1) will contain the product of three such functions and, consequently, will be equal to zero. In the same problem, as is the case generally in problems with cylindrical symmetry, the normal perturbations depend on the angle ϕ which is measured around the axis of symmetry as $\cos m\phi$ or $\sin m\phi$, $m = 0, 1, 2, \ldots$. In the Taylor problem m turns out to be equal to zero for the perturbation which disturbs the steady motion. But if in some problem the stability were disturbed for $m \neq 0$, the integral (6.1) would then contain three factors of the form $\cos m\phi$ or $\sin m\phi$, and a singular point would also be obtained.

We shall show that if Equation (6.1) is satisfied at the critical point and if solutions of the basic series expanded in integral powers of $(R - R_0)$ exist for this near the critical point, then two new series of steady motions expanded in integral powers of

$$\eta = \pm (R - R_0)^{1/2}$$

appear at this point.

These new solutions must have the form

$$\mathbf{V}(R) = \mathbf{U}(R) + \psi = \mathbf{U}_0 + \eta\psi_1 + \eta^2[\mathbf{U}_1 + \psi_2] + \eta^3\psi_3 + \dots \quad (6.2)$$
$$\psi_n|_s = 0, \qquad \operatorname{div} \psi_n = 0$$

and they must satisfy equations analogous to (1.3). For the terms of first order in η we obtain

$$L(\psi_1;\ \mathbf{U}_0) = 0 \quad (6.3)$$

Hence

$$\psi_1 = a_1\mathbf{u}_0(R_0) \quad (6.4)$$

The equations for the terms of second order will be

$$L(\psi_2;\ \mathbf{U}_0) = -(\psi_1 \cdot \nabla)\psi_1 = -a_1^2(\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 \quad (6.5)$$

and their solvability condition

$$\int \mathbf{v}_0 \cdot (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 dV = 0$$

is automatically satisfied by virtue of (6.1). Their solution will have the form

$$\psi_2 = a_1^2\chi_2 + a_2\mathbf{u}_0 \quad (6.6)$$

where a_1 remains undetermined and a_2 is a new unknown constant. Finally, in the third order we obtain the equations

$$L[\psi_3;\ \mathbf{U}_0] = -(\psi_1 \cdot \nabla)[\mathbf{U}_1 + \psi_2] - [(\mathbf{U}_1 + \psi_2) \cdot \nabla]\psi_1 \quad (6.7)$$

into the right-hand side of which Expressions (6.4) and (6.6) should be substituted. Their solvability condition (orthogonality of the right-hand side to $\mathbf{v}_0$) will be

$$a_1^3\int \mathbf{v}_0[(\mathbf{u}_0\cdot\nabla)\chi_2 + (\chi_2 \cdot \nabla)\mathbf{u}_0]\,dV + \quad (6.8)$$
$$+ a_1\int \mathbf{v}_0 \cdot [(\mathbf{u}_0 \cdot \nabla)\mathbf{U}_1 + (\mathbf{U}_1 \cdot \nabla)\mathbf{u}_0]\,dV + 2a_1a_2\int \mathbf{v}_0 \cdot (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 dV = 0$$

The last of the summed terms will be here equal to zero by virtue of (6.1), and a nonzero solution for a_1 will be

$$a_1^2 = -\frac{\int \mathbf{v}_0 \cdot [(\mathbf{u}_0 \cdot \nabla)\, \mathbf{U}_1 + (\mathbf{U}_1 \cdot \nabla)\, \mathbf{u}_0]\, dV}{\int \mathbf{v}_0 \cdot [(\mathbf{u}_0 \cdot \nabla)\, \chi_2 + (\chi_2 \cdot \nabla)\, \mathbf{u}_0]\, dV} \tag{6.9}$$

etc. Thus, at R_0 solutions of the form

$$\mathbf{V} = \mathbf{U} \pm a_1 (R - R_0)^{1/2} \mathbf{u}_0 + \dots \tag{6.10}$$

appear.

It is natural to call such a kind of critical point a branch point. It is easy to show that the least decrements of the motions (6.10) will be $\lambda_0 = \pm\, c(R - R_0)^{1/2} + \dots$, therefore only one of these motions is stable.

Just such a case was experimentally investigated by Taylor [5]. The fluid was between two very long coaxial cylinders which were rotating with equal angular velocity, and the torque operating through the fluid on one of the cylinders was measured. In this case the torque was strictly proportional to R for the basic flow. On the same experimental curve, which gives the torque as a function of R, a break is clearly seen at the critical point R_0. An additional torque which appears above the critical point apparently is actually proportional to $(R - R_0)^{1/2}$. It should be kept in mind that Taylor's cylinders were of finite length so that, strictly speaking, the critical point should be regular. The second steady solution and the additional torque with it also must differ from the basic one by a quantity which is proportional to $(R - R_0)$, but with a very large coefficient of proportionality. It is difficult to distinguish such a curve from the parabola (6.10). There exists a qualitative difference between a regular point and a branch point: at a branch point the second solution exists only for $R > R_0$, whereas at a regular point it is also possible for $R < R_0$ although it is unstable there. If decreasing the Reynolds number, one passes carefully through a regular critical point, it is then possible to retain the second flow for $R < R_0$ also. At a branch point this is absolutely impossible. In the experiments of Lewis apparently, just a regular point was observed, because he says that "when the velocity was gradually decreased, vortices remained until this velocity took a value smaller than that for which they appeared" [4].

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